Pre-Calculus 30

Unit 5 – Exponential and Logarithmic Functions

PC30.9 *- Demonstrate an understanding of logarithms including: evaluating logarithms, relating logarithms to exponents, deriving laws of logarithms, solving equations, graphing.*

** Adapted from Chapter 5 Pearson Pre-Calculus 12*

Name: ______________________
Date: ________________________

Key Terms

Exponential Function – A function of the form $y = c(a)^x, where\ c\ \neq 0, a > 0$
Transformation of Exponential Functions – $y - k = ca^{d(x-h)}, a > 0, c \neq 0, d \neq 0$
$|c|$ – Vertical stretch or compression, $\frac{1}{|d|}$ – Horizontal stretch or compression, $c < 0$ – Reflected in the x-axis (flipped vertically), $d < 0$ – Reflected in the y-axis (flipped horizontally), k – Vertical translation, h – Horizontal Translation
Exponential Function – The general translation of (x, y) corresponds to $(\frac{x}{d} + h, cy + k)$
Natural Logarithm – A logarithm to the base of e where $y = log_e(x) or\ \ y = \ln(x)$

Product of Powers – Multiplying variables with exponents: $x^a \cdot x^b = x^{a+b}$
Quotient of Powers – Dividing variables with exponents: $\frac{x^a}{x^b} = x^{a-b}$
Power of a Power – Variable with an exponent to the power of an exponent: $(x^a)^b = x^{a \cdot b}$
Power of a Quotient – A fraction raised to an exponent: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$
Negative Exponent – An exponent that is negative becomes positive with the reciprocal of the base: $x^{-a} = \frac{1}{x^a}$
Fractional Exponent – A fractional exponent can be written as a radical: $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
Exponential Equation – Contains a power with a variable in the exponent.
Compound Interest – The interest that is earned or paid on both the principal and the accumulated interest.
Compounding Period – The time over which interest is determined; Annually **–** Once per year (1), Semi-Annually – Twice per year (2), Quarterly – Four times a year (4), Monthly – Once a month (12), Semi-Monthly – Twice a month (24), Weekly – Once a week (52), Bi-Weekly – Every other week (26), Daily – Each day (365)
Principal – The initial amount that money is invested or loaned.
Exponential Growth – A function that models exponential growth has the form of $y = ak^{bx}$, where $k^b > 1$, and $a \in R, b \in R, k > 0$. K is the growth factor.
Compound Interest – $A = P\left(1 + \frac{i}{n}\right)^{nt}$, where A is future value, P is the principle (amount invested, or A_0), i is interest rate, n is compounding periods and t is the term.
Exponential Decay – A function that models exponential decay has the form of $y = ak^{bx}$, where $0 < k^b < 1$, and $a \in R, b \in R, k > 0$. K is the decay factor.
Logarithmic Function – The logarithm of a number is an exponent. $log_b c = a$ is the power to which b is raised to get c. The base of the logarithm is the same as the base of the power. When $log_b c = a$, then $c\ = b^a$, where $b > 0$, $b \neq 1$, $c > 0$.
Common Logarithm – Numbers based on powers of 10, $log_{10}x$ is called the common logarithm. When logarithms to base 10 are written, the base is often not shown; so $log_{10}x$ is written as $log\ x$.
Rewriting Logarithms and Exponential Equation – $\mathrm{y} = \log_b x$ is equivalent to $\mathrm{x} = \mathrm{b}^y$
Product Law: $log_b xy = log_b x + log_b y,\ \ b > 0, b \neq 1$
Quotient Law: $log_b\left(\frac{x}{y}\right) = log_b x - log_b y,\ \ b > 0, b \neq 1$
Power Law: $log_b x^k = klog_b x,\ \ b > 0, b \neq 1, k \in R$
Changing Base of a Logarithm – $log_b x = \frac{\log\ \ x}{\log\ \ b}, where\ a, b > 0, b \neq 1; x > 0$
Transformation of Logarithmic Functions – $y - k = clog_a d(x - h), a > 0, c \neq 0, d \neq 0$; $|c|$ – Vertical stretch or compression, $\frac{1}{|d|}$ – Horizontal stretch or compression, $c < 0$ – Reflected in the x-axis (flipped vertically), $d < 0$ – Reflected in the y-axis (flipped horizontally), k – Vertical translation, h – Horizontal Translation
Logarithmic Function – The general translation of (x, y) corresponds to $(\frac{x}{d} + h, cy + k)$

Logarithmic Equation – An equation that contains the logarithm of a variable. The laws of logarithms may be used to solve logarithmic equations

Annuity – A continual payment of a fixed amount. Solved using the formula: $A = P\left[\frac{\left(1+\frac{i}{n}\right)^{nt} - 1}{\frac{i}{n}}\right]$, where *A* is the future value, *P* is the payment amount, *i* is the rate, *n* is the compounding periods, *t* is the term length.
Richter Scale – The intensity of vibrations of an earthquake, I microns, is measured 100 km away from the epicentre of the earthquake. The intensity is compared to the intensity, S, of a standard earthquaie. The logarithmic scale for measuring the intensity is: $M = log\left(\frac{I}{S}\right)$

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Unit Checklist

5.1 – Graphing Exponential Functions
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5.2 – Analyzing Exponential Functions
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5.1 – Graphing Exponential Functions

Exponential Function – A function of the form $y = c(a)^x, where\ c\ \neq 0, a > 0$

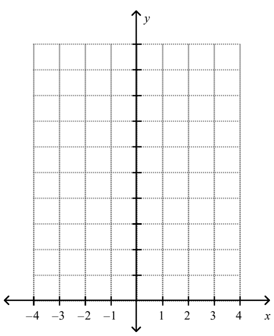
Example 1: a) Fill in the table of values, sketch each graph, and determine the domain and range.

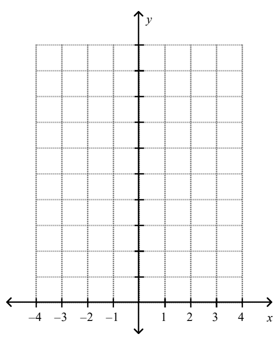
a) $y = 10^x$	
-3	
-2	
-1	
0	
1	
2	
3	

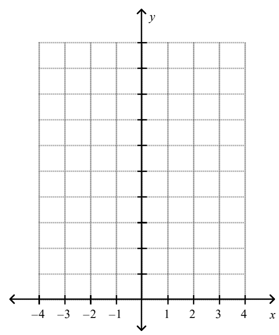
b) $y = 2(5)^x$	
-3	
-2	
-1	
0	
1	
2	
3	

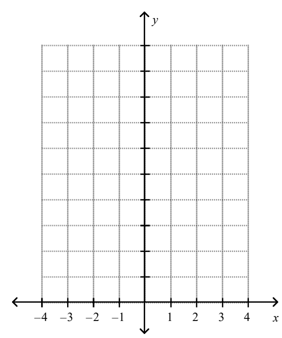
c) $y = \left(\frac{1}{2}\right)^x$	
-3	
-2	
-1	
0	
1	
2	
3	

d) $y = 8\left(\frac{1}{4}\right)^x$	
-3	
-2	
-1	
0	
1	
2	
3	





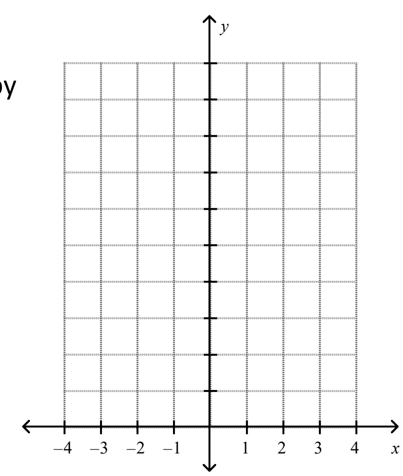




Example 2: Determine, from the following table of values, that the function is exponential:

x	y
-3	0.625
-2	1.25
-1	2.5
0	5
1	10
2	20
3	40

Example 3: Graph $= \left(\frac{1}{3}\right)^x$, determine the following: the effect on y when x increases by 1, increasing/decreasing, the intercepts, the equation of any asymptotes, domain, and range.



If you want another example of this go to Example 1 on page 345.

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5.2 – Analyzing Exponential Functions

Transformation of Exponential Functions – $y - k = ca^{d(x-h)}, a > 0, c \neq 0, d \neq 0$
$|c|$ – Vertical stretch or compression
$\frac{1}{|d|}$ – Horizontal stretch or compression
$c < 0$ – Reflected in the x-axis (flipped vertically)
$d < 0$ – Reflected in the y-axis (flipped horizontally)
k – Vertical translation
h – Horizontal Translation

Exponential Function – The general translation of (x, y) corresponds to $(\frac{x}{d} + h, cy + k)$

Graphing Exponential Functions with Transformations

1) Determine lattice points for the original function
2) Apply the transformations to your x and y values
3) Plot the new points

Example 1: a) Sketch the graph of $y = 2^x$. Determine whether the function is increasing or decreasing, the intercepts, equation of any asymptotes, domain and range.

b) Use the graph of $y = 2^x$ to sketch the graph of $y = 3(2^{-x+2})$. Determine whether the function is increasing or decreasing, the intercepts, equation of any asymptotes, domain and range.

If you want another example of this go to Example 2 on page 347.

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Natural Logarithm – A logarithm to the base of e where $y = log_e(x) or\ \ y = \ln(x)$

$$\log(a)^x = x(loga)$$

Solving for x in Exponential Functions

1) Substitute your known value in for y
2) Isolate your base and exponent by dividing both sides by a
3) Take the log of both sides
4) Isolate your x value by arranging knowing that $\log(a)^x = x(\log a)$

Example 2: The temperature of cooling water in a cup can be modelled by the equation: $t = 89.726(0.972)^m$, where m is the number of minutes that have passed and t is the temperature in °C. Using this determine:

a) The temperature after 52 minutes

b) The time when the water was 51°

Example 3: For every metre below the surface of water, the light intensity is reduced by 2.5%. The percent, P, of light remaining at a depth d metres can be modelled by the function: $P = 100(0.975)^d$

a) Sketch the graph of the function, determine whether the function is increasing or decreasing, the intercepts, equation of any asymptotes, domain and range.

b) To the nearest percent, how much light remains at a depth of 10 m?

c) To the nearest metre, what is the depth when only 50% of the light remains?

If you want another example of this go to Example 3 on page 348

5.3 – Solving Exponential Equations

Product of Powers – Multiplying variables with exponents: $x^a \cdot x^b = x^{a+b}$

Quotient of Powers – Dividing variables with exponents: $\frac{x^a}{x^b} = x^{a-b}$

Power of a Power – Variable with an exponent to the power of an exponent: $(x^a)^b = x^{a \cdot b}$

Power of a Quotient – A fraction raised to an exponent: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Negative Exponent – An exponent that is negative becomes positive with the reciprocal of the base: $x^{-a} = \frac{1}{x^a}$

Fractional Exponent – A fractional exponent can be written as a radical: $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Exponential Equation – Contains a power with a variable in the exponent.

Solving Exponential Equations Using Common Bases

1) Determine the common base on the right side and left side of the equation
2) Divide your number by the common base, counting how many times you've done this, until you get to 1
3) Rewrite this number as a common base to the power of the amount of times you were able to divide
4) Make sure the common bases are the same, then you can cancel them out, leaving only the exponents
5) Solve for x

Example 1: Solve each equation:

a) $4^x = \frac{1}{256}$

b) $27^x = 9^{2x-1}$

If you want another example of this go to Example 1 on page 359.

Example 2: Solve each equation:

a) $2^x = 8\sqrt[3]{2}$

b) $\left(\sqrt{125}\right)^{2x+1} = \sqrt[3]{625}$

If you want another example of this go to Example 2 on page 360.

Compound Interest – *The interest that is earned or paid on both the principal and the accumulated interest.*
Compounding Period – *The time over which interest is determined*

Number of Compounding Periods in a Year					
Annually	**Semi-annually**	**Quarterly**	**Monthly**	**Weekly**	**Daily**

Principal – The initial amount that money is invested or loaned.
Exponential Growth – A function that models exponential growth has the form of $y = ak^{bx}$, where $k^b > 1$, and $a \in R, b \in R, k > 0$. K is the growth factor.

Compound Interest – $A = P\left(1 + \frac{i}{n}\right)^{nt}$, where A is future value, P is the principle (amount invested, or A_0), i is interest rate, n is compounding periods and t is the term.

Example 3: A principal of $1500 is invested at 4% annual interest, compounded quarterly. To the nearest quarter of a year, when will the amount be $2500?

If you want another example of this go to Example 3 on page 362.

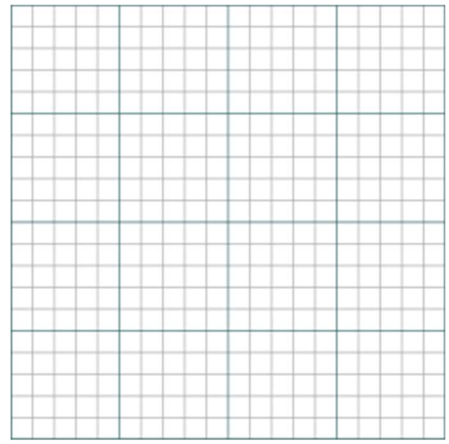
Exponential Decay – A function that models exponential decay has the form of $y = ak^{bx}$, where $0 < k^b < 1$, and $a \in R, b \in R, k > 0$. K is the decay factor.

Example 4: The function $P = 101.3(0.88)^h$ models the atmospheric pressure, P kilopascals, at an altitude of h kilometres. If the cabin pressure in an airplane is less than 70 kPa, passengers can suffer from altitude sickness. To the nearest kilometre, at what altitude is the atmospheric pressure 70 kPa?

If you want another example of this go to Example 4 on page 363.

5.4 – Logarithms and the Logarithmic Function

Example 1: Graph the functions $y = 10^x$ and $y = log_{10}x$. Determine whether the function is increasing or decreasing, the intercepts, equation of any asymptotes, domain and range of each function. How are these graphs related?



a) $y = 10^x$	
− 2	
− 1	
0	
1	
2	

b) $y = log_{10}x$	
− 1	
0	
0.01	
0.1	
1	
10	
100	

Logarithmic Function – The logarithm of a number is an exponent. $log_b c = a$ is the power to which b is raised to get c. The base of the logarithm is the same as the base of the power. When $log_b c = a$, then $c = b^a$, where $b > 0, b \neq 1, c > 0$.

Common Logarithm – Numbers based on powers of 10, $log_{10}x$ is called the common logarithm. When logarithms to base 10 are written, the base is often not shown; so $log_{10}x$ is written as $log\ x$.

$$\mathrm{y} = \log_b x \text{ is equivalent to } \mathrm{x} = \mathrm{b}^y$$

Changing Form Between Exponential and Logarithmic

1) Determine the base, rewrite as the new base
2) Invert the x and y values

Example 2: Write each exponential expression as a logarithmic expression:

a) $3^3 = 27$ b) $5^{-2} = \frac{1}{25}$ c) $4^0 = 1$

Write each logarithmic expression as an exponential expression:

a) $log_3 81 = 4$ b) $log_5 125 = 3$ c) $log_6 1 = 0$

If you want another example of this go to Example 1 on page 377.

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Evaluating Logarithms

1) Rewrite as an exponential equation
2) Simplify

Example 3: Evaluate each logarithm:

a) $log_5 3125$

b) $log_6\left(\frac{1}{216}\right)$

c) $log_8\left(2\sqrt[3]{2}\right)$

If you want another example of this go to Example 2 on page 378.

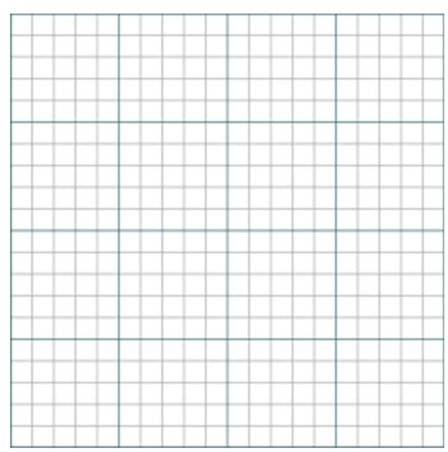
Example 4: To the nearest tenth, estimate the value of $log_5 100$.

If you want another example of this go to Example 3 on page 379.

Sketching Graphs of Logarithmic Functions

1) Determine the inverse Exponential Function
2) Create a table of values for the exponential function
3) Invert your x and y values to sketch the points

Example 5: Graph $y = log_4 x$. Identify the intercepts, the equations of any asymptotes, and the domain and range of the function.



If you want another example of this go to Example 4 on page 379.

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5.5 – The Laws of Logarithm

Product Law: $log_b xy = log_b x + log_b y,\ \ b > 0, b \neq 1$
Quotient Law: $log_b \left(\frac{x}{y}\right) = log_b x - log_b y,\ \ b > 0, b \neq 1$
Power Law: $log_b x^k = k log_b x,\ \ b > 0, b \neq 1, k \in R$

Example 1: Use a law of logarithms to simplify each expression. Use a calculator to verify the answers:
a) $\log 7 + \log 8$ b) $5 \log 2$ c) $\log 80 - \log 16$

If you want another example of this go to Example 1 on page 390.

Example 2: Write each expression as a single logarithm
a) $\log x + 3 \log y$ b) $\log x + 2 \log y - 4 \log z$ c) $log_2 6 - 3$

If you want another example of this go to Example 2 on page 390.

Example 3: Write each expression in terms of log a, log b, and/or log c.
a) $log\left(\frac{a}{b^2}\right)$ b) $log\left(\frac{a^2 b^{\frac{1}{3}}}{c}\right)$

If you want another example of this go to Example 3 on page 391.

Example 4: Evaluate each expression:
a) $3 log_9 6 - log_9 72$ b) $2 log_4 6 - 3 log_4 3 + log_4 12$

If you want another example of this go to Example 4 on page 392.

5.6 – Analyzing Logarithmic Functions

Example 1: Solve for y in $y = log_b x$

Changing Base of a Logarithm – $log_b x = \frac{\log}{\log\ \ b}, where\ a, b > 0, b \neq 1; x > 0$

Example 2: Approximate the value of each logarithm, to the nearest thousandth. Write the related exponential expression.

a) $log_5 50$

b) $log_8 6$

If you want another example of this go to Example 1 on page 402.

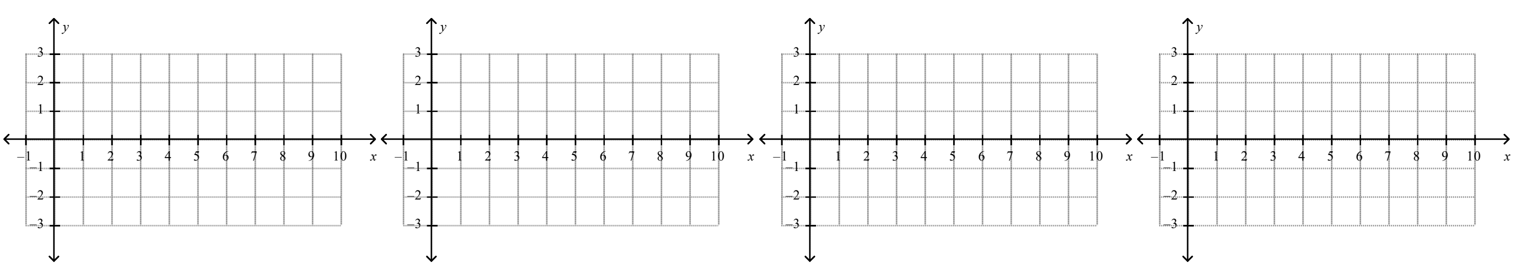
Example 3: Fill in the table of values, sketch each graph, and determine the domain and range.

$y = log_{10}x$	
-1	
0	
0.5	
1	
10	

$y = 2log_{10}x$	
-1	
0	
0.5	
1	
10	

$y = -3log_{10}x$	
-1	
0	
0.5	
1	
10	

$y = log_8 x$	
-1	
0	
0.5	
1	
8	



Transformation of Logarithmic Functions – $y - k = clog_a d(x - h), a > 0, c \neq 0, d \neq 0$

$|c|$ – Vertical stretch or compression

$\frac{1}{|d|}$ – Horizontal stretch or compression

$c < 0$ – Reflected in the x-axis (flipped vertically)

$d < 0$ – Reflected in the y-axis (flipped horizontally)

k – Vertical translation

h – Horizontal Translation

Logarithmic Function – The general translation of (x, y) corresponds to $(\frac{x}{d} + h, cy + k)$

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Sketching Graphs of Logarithmic Functions

1) Use lattice points for the x-intercept and (a,1)
2) Determine any transformations made
3) Transform the points

Example 4: Sketch the graph of $y = log_3 x$ and $y = log_3(2x + 6)$

Example 5: Sketch the graph of $y = log_2 x$ and $y = log_2 2x - 1$

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5.6 – Analyzing Logarithmic Functions

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Example 6: Sketch the graph of $y = log_5 x$ and $y = -\frac{1}{2}log_5(2x + 2)$

Example 7: Sketch the graph of $y = log_2 x$ and $y = 2log_2\left(-\frac{1}{3}(x + 1) + 2\right)$

5.7 – Solving Logarithmic and Exponential Functions

Logarithmic Equation – An equation that contains the logarithm of a variable. The laws of logarithms may be used to solve logarithmic equations.

Example 1: Solve: $log_3 9x + log_3 x = 4$. Verify the solution.

If you want another example of this go to Example 1 on page 418.

Example 2: Solve then verify each equation.

a) $\log 6x = \log(x + 6) + \log(x - 1)$

b) $3 = log_2(x + 2) + log_2 x$

If you want another example of this go to Example 2 on page 419.

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Example 3: Solve each exponential equation algebraically. Give the solution to the nearest hundredth.

a) $4^x = 12$

b) $3(2^{x+1}) = 36$

c) $3^{x+1} = 6^x$

d) $2^{x+3} = 6^{x-1}$

If you want another example of this go to Example 3 on page 420.

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5.8 – Solving Problems with Exponents and Logarithms

Annuity – A continual payment of a fixed amount. Solved using the formula:

Investments: $A = P\left[\frac{\left(1+\frac{i}{n}\right)^{nt} - 1}{\frac{i}{n}}\right]$ Loans: $A = P\left[\frac{1-\left(1+\frac{i}{n}\right)^{-nt}}{\frac{i}{n}}\right]$

Example 1: Determine how many monthly investments of $200 would have to be made into an account that pays 6% annual interest, compounded monthly, for the future value to be $100 000.

If you want another example of this go to Example 1 on page 431.

Example 2: A person borrows $15 000 to buy a car. The person can afford to pay $300 a month. The loan will be repaid with equal monthly payments at 6% annual interest, compounded monthly. How many monthly payments will the person make?

If you want another example of this go to Example 2 on page 432.

Richter Scale – The intensity of vibrations of an earthquake, I microns, is measured 100 km away from the epicentre of the earthquake. The intensity is compared to the intensity, S, of a standard earthquake. The logarithmic scale for measuring the intensity is: $M = log\left(\frac{I}{S}\right)$

Example 3: The most intense earthquake ever recorded was in Chile in May 1960, with a magnitude of 9.5. In January of 2010, Haiti experienced an earthquake with a magnitude of 7.0. Calculate the intensity of the earthquake in Chile and Haiti. How many times as intense as the Haiti earthquake was the Chile earthquake?

If you want another example of this go to Example 2 on page 432.