Pre-Calculus 30

Unit 3 – Transforming Graphs of Functions

PC30.7 - *Extend understanding of transformations to include functions (given in equation or graph form) in general, including horizontal and vertical translations, and horizontal and vertical stretches.*

PC30.8 - *Demonstrate understanding of functions, relations, inverses and their related equations resulting from reflections through the: x-axis, y-axis and line y = x*

** Adapted from Chapter 3 Pearson Pre-Calculus 12*

Key Terms

Translation - to describe a function that moves an object a certain distance. The object is not altered in any other way.
Horizontal Translation – The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$. The graph will be translated h units left or right. The point (x, y) on $y = f(x)$ corresponds to the point $(x + h, y)$ on $y = f(x - h)$.
Vertical Translation – The graph of $y - k = f(x)$ is a vertical translation of the graph of $y = f(x)$. The graph will be translated k units up or down. The point (x, y) on $y = f(x)$ corresponds to the point $(x, y + k)$ on $y - k = f(x)$.
Explicit Equation – an equation that is written in terms of the independent variable.
Lattice Point – a point at the intersection of two or more grid lines.
Reflection - a type of transformation in which the function is flipped across a line of reflection to create a new function. Each point of the function is the same distance from the reflection line as the original function is.
Reflecting in the x-axis – the graph of $y = -f(x)$ is the image of the graph $y = f(x)$ after a reflection in the x-axis. A point (x, y) on $y = f(x)$ would correspond to the point $(x, -y)$ on $y = -f(x)$.
Reflecting in the y-axis – the graph of $y = f(-x)$is the image of the graph of $y = f(x)$ after a reflection in the y-axis. The point (x, y) on $y = f(x)$ would correspond to the point $(-x, y)$ on $y = f(-x)$.
Vertical Stretch, Compressions or Reflections – When the graph of $y = af(x)$ is the image of the graph of $y = f(x)$. The point (x, y) on $y = f(x)$ corresponds to the point (x, ay) on $y = af(x)$.
Vertical Stretch – In the graph $y = af(x)$, if $|a| > 1$, then the graph of $y = f(x)$ has been vertically stretched by a factor of $|a|$.
Vertical Compression – In the graph $y = af(x)$, if $0 < |a| < 1$, then the graph of $y = f(x)$ has been vertically compressed by a factor of $|a|$.
Vertical Reflection – In the graph $y = af(x)$, if $a < 0$, the graph of $y = f(x)$ has a reflection in the x-axis as well as a possible stretch or compression.
Horizontal Stretch, Compressions or Reflections – When the graph of $y = bf(x)$ is the image of the graph of $y = f(x)$. The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, y\right)$ on $y = f(bx)$.
Horizontal Stretch – In the graph $y = f(bx)$, if $0 < |b| < 1$, then the graph of $y = f(x)$ has been horizontally stretched by a factor of $\frac{1}{|b|}$.
Horizontal Compression – In the graph $y = f(bx)$, if $|b| > 1$, then the graph of $y = f(x)$ has been horizontally compressed by a factor of $\frac{1}{|b|}$.
Horizontal Reflection – In the graph $y = f(bx)$, if $b < 0$, the graph of $y = f(x)$ has a reflection in the y-axis as well as a possible stretch or compression.
Stretch, Compression and Reflection – The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, ay\right)$ on $y = af(bx)$.
Combining Transformations = $y - k = af(b(x - h))$ is the image of the graph of $y = f(x)$ after the transformations: horizontal stretch or compression by the factor of $\frac{1}{|b|}$, reflection on the y-axis if $b < 0$, vertical stretch or compression by the factor of $|a|$, reflection on the x-axis if $a < 0$, horizontal translation of h units, vertical translation of k units. Point (x, y) on the graph y = f(x) corresponds to the point $\left(\frac{x}{b} + h,\ ay + k\right)$ on the graph $y - k = af(b(x - h))$.
Invariant Points – Points that do not change and will be on both f(x) and $\sqrt{f(x)}$
Inverse – Opposite or "reverse".
Reflecting in the Line $y = x$ – For a function $y = f(x)$, the graph of $x = f(y)$ is the image of the graph of $y = f(x)$ after a reflection in the line $y = x$. A point (x, y) on $y = f(x)$ corresponds to the point (y, x) on the graph of $x = f(y)$.

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Unit Checklist

3.1 – Transforming Graphs of Functions - Translations
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3.2 – Reflecting Graphs of Functions
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3.3 – Stretching and Compressing Graphs of Functions
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Unit Quiz
Review – Page 213 – All (answers on page 218)

3.4 – Combining Transformations of Functions - Part 1
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3.4 – Combining Transformations of Functions – Part 2
Page 229 #8, 9, 10, 11, 12 (answers on page 233)

3.5 – Inverse Relations – Part 1
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Unit Test

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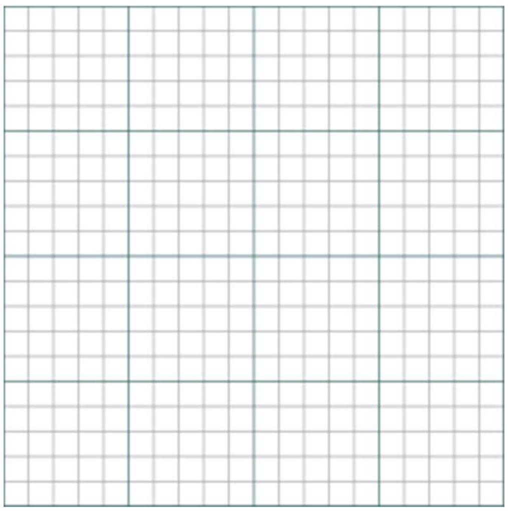
3.1 – Transforming Graphs of Functions – Translations

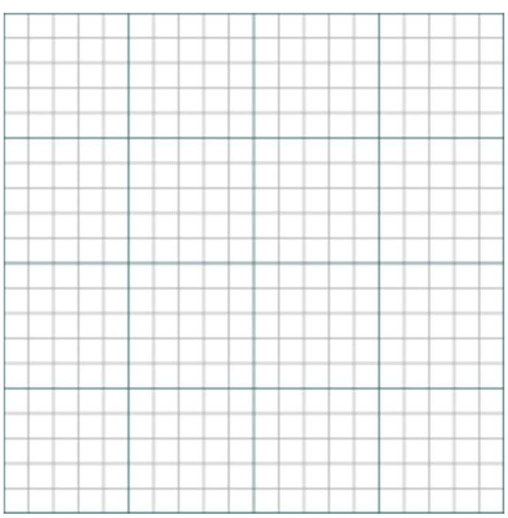
Translation - to describe a function that moves an object a certain distance. The object is not altered in any other way.

Example 1: Graph the function $f(x) = x^2 + k$, let k = 0, 3, and – 3. Use a table of values and state the domain and range.



Vertical Translation – The graph of $y - k = f(x)$ is a vertical translation of the graph of $y = f(x)$. The graph will be translated k units up or down. The point (x, y) on $y = f(x)$ corresponds to the point $(x, y + k)$ on $y - k = f(x)$.

Example 2: Graph the function $f(x) = (x - h)^2$, let h = 0, 4, and – 3. Use a table of values, and state the domain and range.



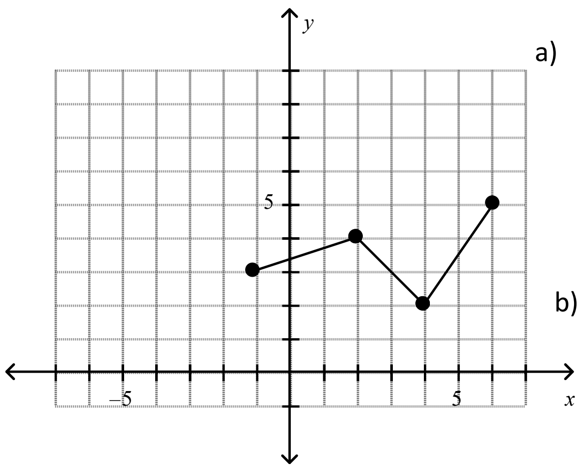
Horizontal Translation – The graph of $y = f(x - h)$ is a horizontal translation of the graph of $y = f(x)$. The graph will be translated h units left or right. The point (x, y) on $y = f(x)$ corresponds to the point $(x + h, y)$ on $y = f(x - h)$.

Lattice Point – a point at the intersection of two or more grid lines.

Graphing with Horizontal and/or Vertical Translations

1. Choose lattice points on the graph
2. Apply the translations to these points
3. Plot the translated point on the graph.
4. Sketch

Example 3: Here is the graph of $y = g(x)$. Sketch the image graph after each translation. Write the equation of the image graph In terms of the function g. State the domain and range of each function.

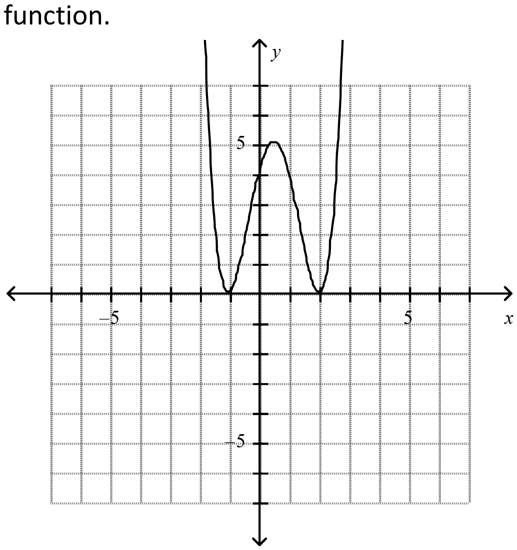


a) A translation of 4 units left

b) A translation of 1 unit up

If you want another example of this go to Example 1 on page 165.

Example 4: Here is the graph of $y = f(x)$. Sketch the image graph after a translation of 4 units left and 5 units down. Write the equation of the image graph in terms of the function f. State the domain and range of each function.



If you want another example of this go to example 2 on page 166.

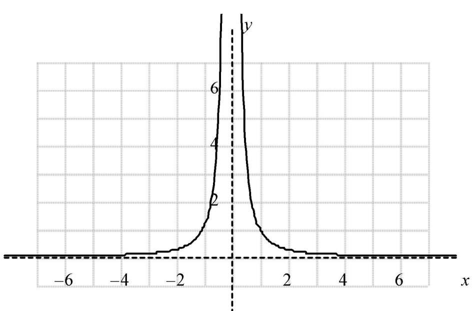
Explicit Equation – an equation that is written in terms of the independent variable.

Example 5: The graph of $y = \frac{1}{x}$ is translated 3 units left and 2 units up. What is the equation of the image graph?

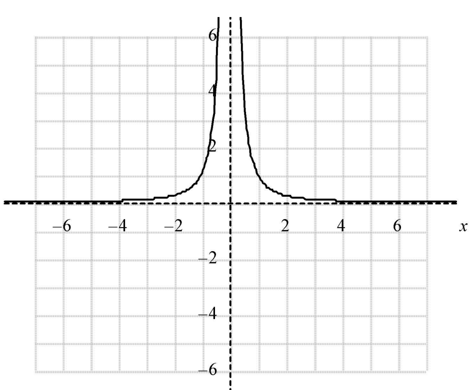
If you want another example of this go to Example 3 on page 167.

Example 6: Describe how the graph of $y = \frac{1}{x^2}$ could have been translated to create the graph of each function below. What are the equations of the asymptotes of each image graph?

a) $y - 3 = \frac{1}{x^2}$



b) $y + 4 = \frac{1}{(x+1)^2}$



If you want another example of this go o example 4 on page 167.

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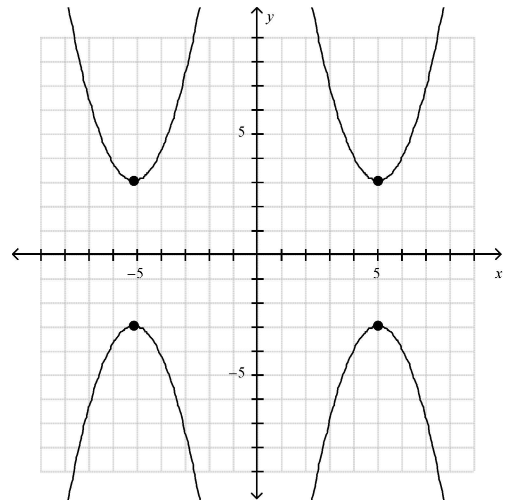
3.2 – Reflecting Graphs of Functions

Reflection - a type of transformation in which the function is flipped across a line of reflection to create a new function. Each point of the function is the same distance from the reflection line as the original function is.

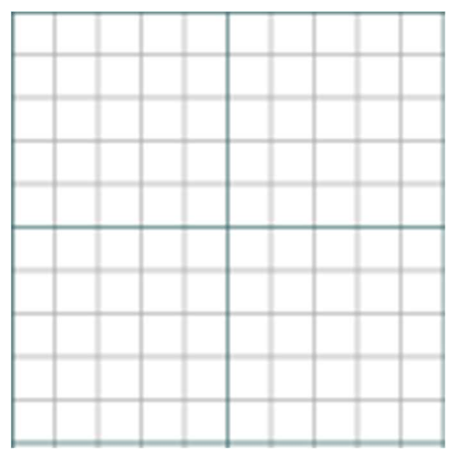
Reflecting in the x-axis – the graph of $y = -af(x)$ is the image of the graph $y = f(x)$ after a reflection in the x-axis. A point (x, y) on $y = f(x)$ would correspond to the point $(x, -y)$ on $y = -af(x)$.

Reflecting in the y-axis – the graph of $y = f(-x)$is the image of the graph of $y = f(x)$ after a reflection in the y-axis. The point (x, y) on $y = f(x)$ would correspond to the point $(-x, y)$ on $y = f(-x)$.



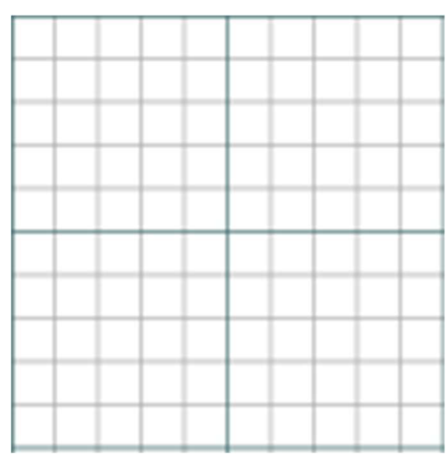
Example 1: For each of the following points label which quadrant they are in, plot the point on the graph, and write the reflected point across the x-axis.

Point	Quadrant	Point Reflected on x-axis	Quadrant
A (2, 4)			
B (-3, 1)			
C (-2, -3)			
D (4, -3)			

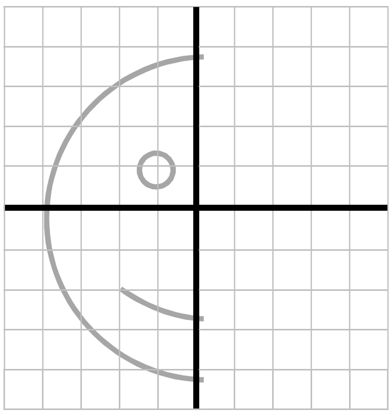


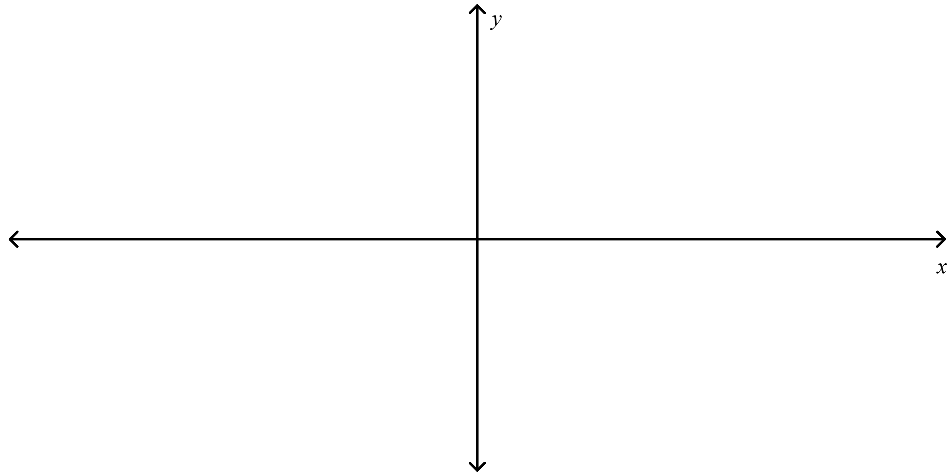
Example 2: For each of the following points label which quadrant they are in, plot the point on the graph, and write the reflected point across the y-axis.

Point	Quadrant	Point Reflected on x-axis	Quadrant
A (2, 4)			
B (-3, 1)			
C (-2, -3)			
D (4, -3)			



Example 3: Reflect the following image across the y-axis., then write your name reflected in each quadrant.





Sketching the Graph of a Polynomial Functions after a Reflection

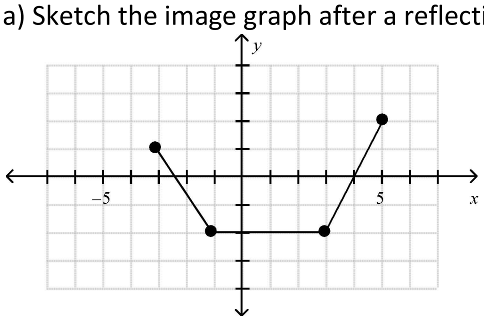
1. Determine lattice points for the graph or estimate coordinates
2. Reflect the points remembering the following:

$$y = f(x) \rightarrow (x, y)$$
$$y = -f(x) \rightarrow (x, -y)$$
$$y = f(-x) \rightarrow (-x, y)$$
$$y = -f(-x) \rightarrow (-x, -y)$$

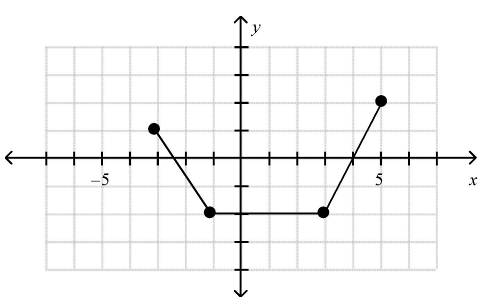
3. Plot the points then sketch.

Example 4: Here is the graph of y = g(x).

a) Sketch the image graph after a reflection in the y-axis. State the domain and range of each function.



b) Sketch the image graph after a reflection in the x-axis. State the domain and range of each function.

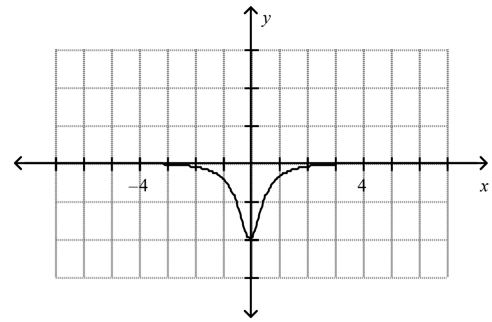


If you want another example of this go to example 1 on page 180.

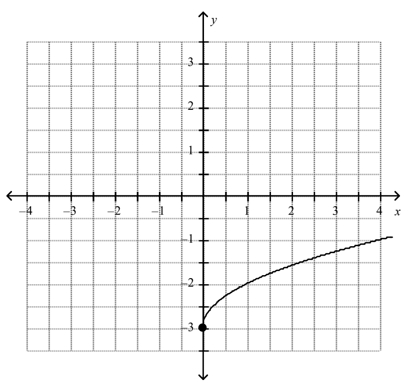
Writing an Equation of a Reflection Image

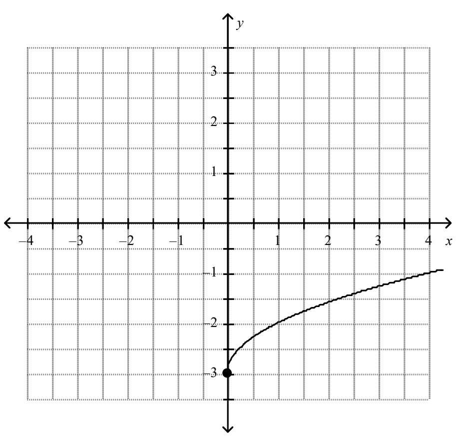
1. Determine if you are reflecting in the x-axis or the y-axis
2. Substitute your x or your y value in as a – 1
3. Simplify

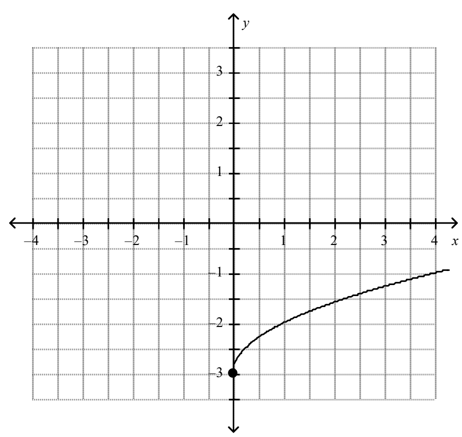
Example 5: The graph of $y = \frac{1}{-2x^2-0.5}$ was reflected in the x-axis and on the y-axis. What is an equation for each of the images?

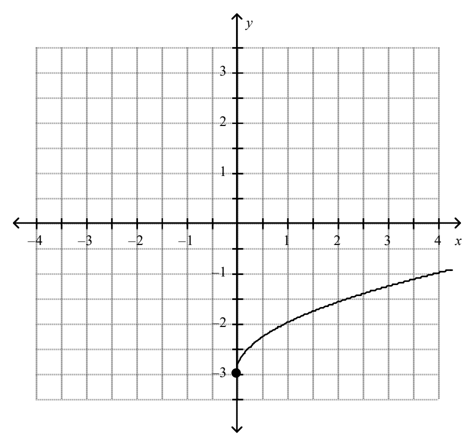


Example 6: Here is the graph of $y = \sqrt{x} - 3$. Fill in a table of values to show the reflection of $y = -f(x)$, $y = f(-x)$ and $y = -f(-x)$. Sketch each function, find the domain and range of each function, and determine an equation for each situation.









If you want another example of this go to Example 2 on page 181.

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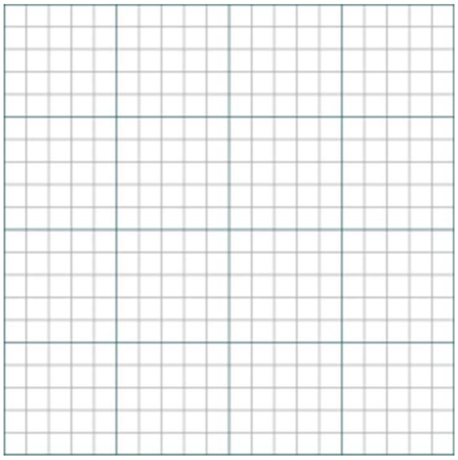
3.3 – Stretching and Compressing Graphs of Functions

Vertical Stretch, Compressions or Reflections – When the graph of $y = af(x)$ is the image of the graph of $y = f(x)$. The point (x, y) on $y = f(x)$ corresponds to the point (x, ay) on $y = af(x)$.

Example 1: Graph the function $f(x) = ax^2$, let a = 1, 2, 0.5, – 2 and – 0.5 . Use a table of values.

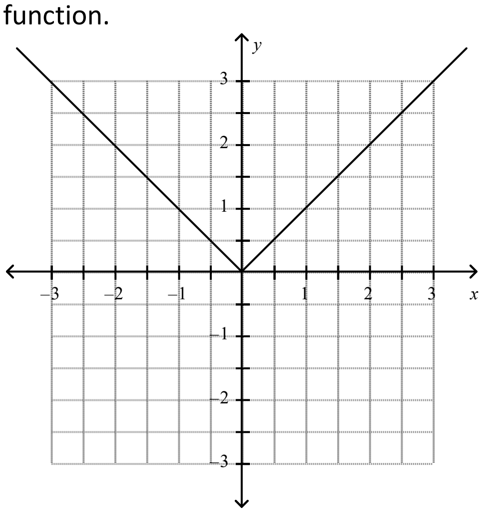


Vertical Stretch – In the graph $y = af(x)$, if $|a| > 1$, then the graph of $y = f(x)$ has been vertically stretched by a factor of $|a|$.
Vertical Compression – In the graph $y = af(x)$, if $0 < |a| < 1$, then the graph of $y = f(x)$ has been vertically compressed by a factor of $|a|$.
Vertical Reflection – In the graph $y = af(x)$, if $a < 0$, the graph of $y = f(x)$ has a reflection in the x-axis as well as a possible stretch or compression.

Sketching the Graph of a Function with a Vertical Stretch and Reflection

1. Choose lattice points on the graph
2. Apply the transformation by the factor of a
3. Plot the new points and sketch

Example 2: Here is the graph of $y = f(x)$. Sketch the graph of $y = -\frac{1}{4}f(x)$. State the domain and range of each function.



If you want another example of this go to example 1 on page 196.

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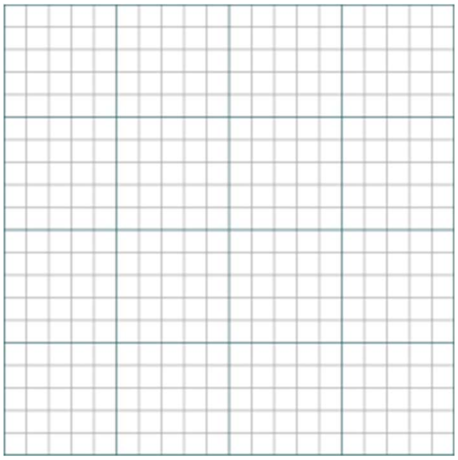
Horizontal Stretch, Compressions or Reflections – When the graph of $y = bf(x)$ is the image of the graph of $y = f(x)$. The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, y\right)$ on $y = f(bx)$.

Example 3: Graph the function $f(x) = \sqrt{bx}$, let a = 1, 2, 0.5, and – 0.5 . Use a table of values.

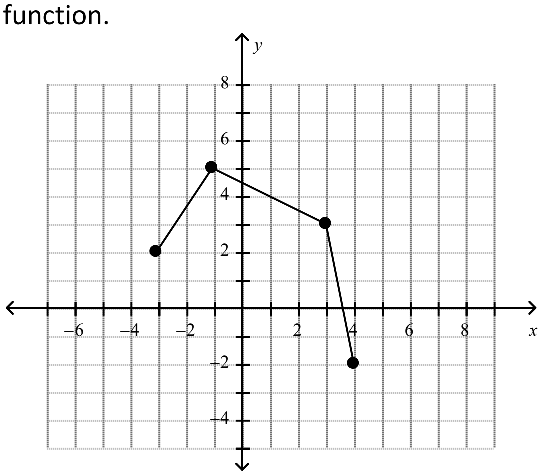


Horizontal Stretch – In the graph $y = f(bx)$, if $0 < |b| < 1$, then the graph of $y = f(x)$ has been horizontally stretched by a factor of $\frac{1}{|b|}$.
Horizontal Compression – In the graph $y = f(bx)$, if $|b| > 1$, then the graph of $y = f(x)$ has been horizontally compressed by a factor of $\frac{1}{|b|}$.
Horizontal Reflection – In the graph $y = f(bx)$, if $b < 0$, the graph of $y = f(x)$ has a reflection in the y-axis as well as a possible stretch or compression.

Sketching the Graph of a Function with a Horizontal Stretch and Reflection

1. Choose lattice points on the graph
2. Apply the transformation by the factor of $\frac{1}{b}$
3. Plot the new points and sketch

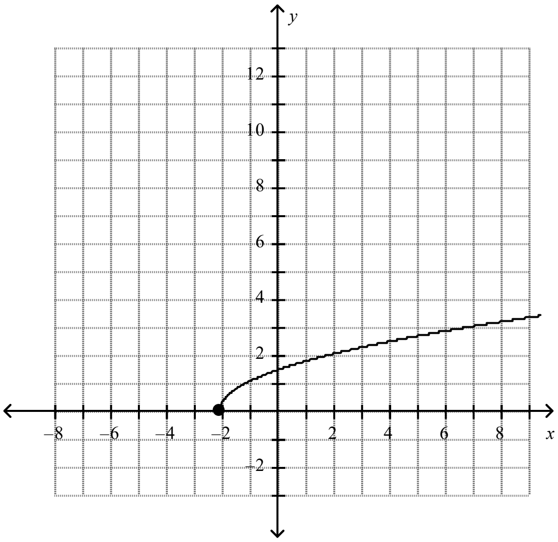
Example 4: Here is the graph of $y = g(x)$. Sketch the graph of $y = g(0.5x)$. State the domain and range of each function.



If you want another example of this go to example 2 on page 198.

Stretch, Compression and Reflection – The point (x, y) on $y = f(x)$ corresponds to the point $\left(\frac{x}{b}, ay\right)$ on $y = af(bx)$.

Example 5: Here is the graph of $y = f(x)$. Sketch the graph of $y = 4f(-0.5x)$. State the domain and range.

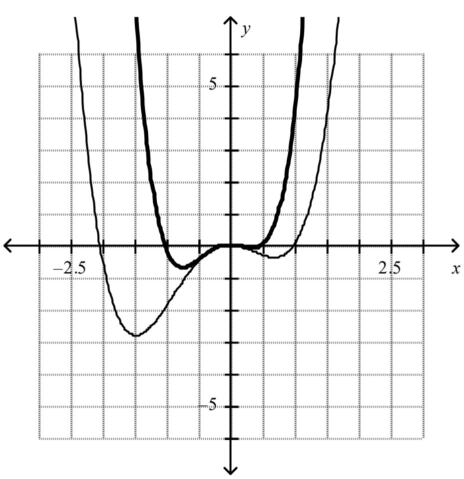


If you want another example of this go to example 3 on page 199.

Determining an Equation After a Transformation

1. Identify corresponding points on both functions, such as intercepts or local maximum/minimums
2. Determine the values of a and b necessary to go from (x, y) to $\left(\frac{x}{b}, ay\right)$
3. Substitute into $y = af(bx)$.

Example 6: The graphs of $y = f(x)$ and its image after a vertical and/or horizontal compression are shown. Write an equation of the image graph in terms of the function f.



If you want another example of this go to example 4 on page 200.

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3.4 – Combining Transformations of Functions – Part 1

Combining Transformations = $y - k = af(b(x-h))$ is the image of the graph of $y = f(x)$ after the transformations:

- Horizontal stretch or compression by the factor of $\frac{1}{|b|}$
- Reflection on the y-axis if $b < 0$
- Vertical stretch or compression by the factor of $|a|$
- Reflection on the x-axis if $a < 0$
- Horizontal translation of h units
- Vertical translation of k units.

Point (x, y) on the graph y = f(x) corresponds to the point $\left(\frac{x}{b} + h,\ ay + k\right)$ on the graph $y - k = af(b(x-h))$.

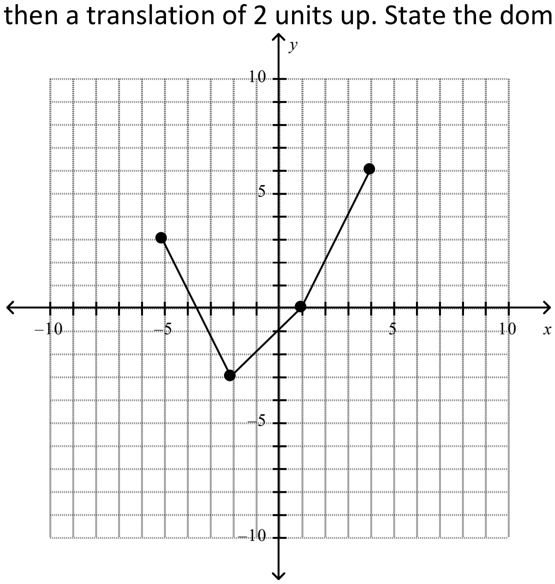
Example 1: a) Describe all the transformations for the following function: $y - 6 = f(4(x+2))$

b) The point (2,4) is on the graph y = f(x), what point will it be on the transformed function?

Sketching Transformations of Graphs

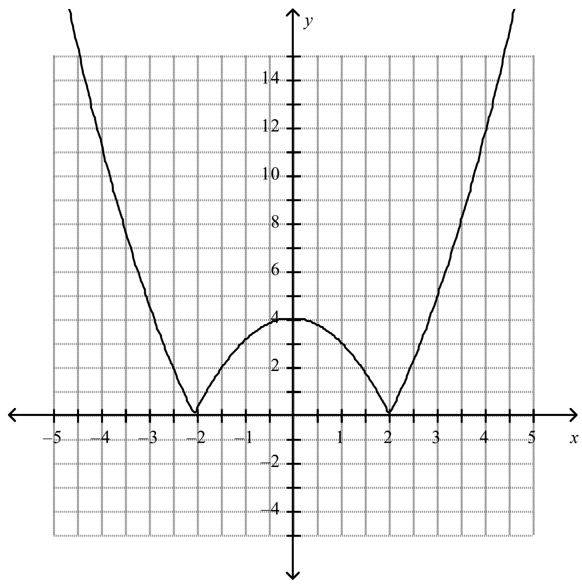
1. Choose lattice points
2. Determine all transformations
3. Apply all stretches and compressions first
4. Apply all reflections
5. Apply all translations

Example 2: Here is the graph of $y = g(x)$. Sketch and label its image after a vertical compression by a factor of $\frac{1}{3}$, then a translation of 2 units up. State the domain of both functions.

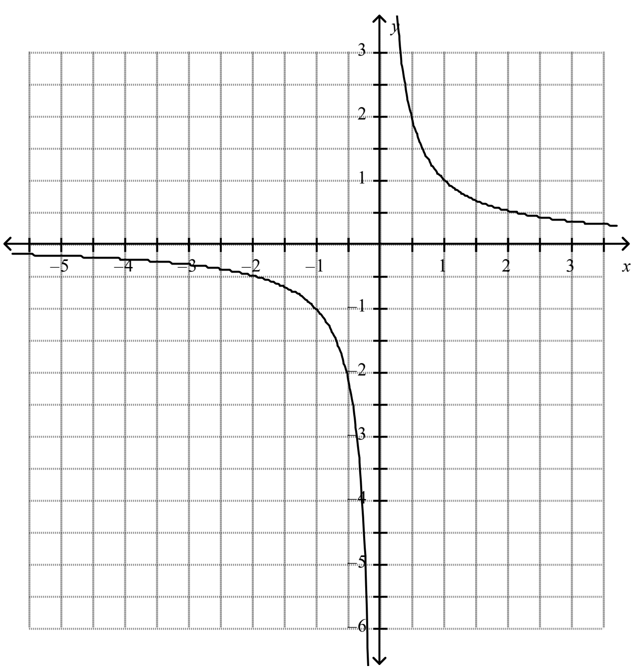


If you want another example of this go to example 1 on page 221.

Example 3: Here is the graph of y = f(x). Sketch the graph of $y - 6 = f(4(x + 2))$. State the domain and range of both functions.



Example 4: Here is the graph of y = f(x). Sketch the graph of $y + 4 = f(\frac{1}{2}(x + 1))$. State the domain and range of both functions.



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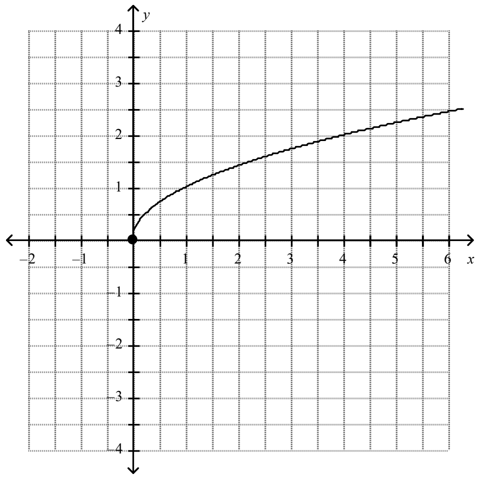
3.4 – Combining Transformations of Functions – Part 2

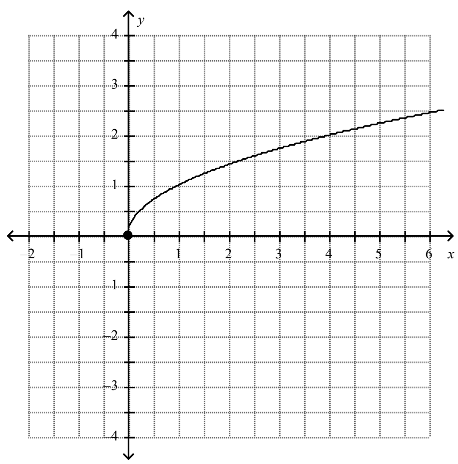
Sketching Transformations of Graphs

1. Choose lattice points
2. Determine all transformations, apply stretches and compressions, then reflections, then translations
3. Find your point $\left(\frac{x}{b}+h,\ ay+k\right)$
4. Fill in table of values of the image of each point, graph

Example 1: Use the graph of $y = \sqrt{x}$ to graph $y - 2 = -\sqrt{3x+3}$. What are the domain and range of the transformed function?



Example 2: Use the graph of $y = \sqrt{x}$ to graph $y + 1 = 2\sqrt{-x+3}$. What are the domain and range of the transformed function?

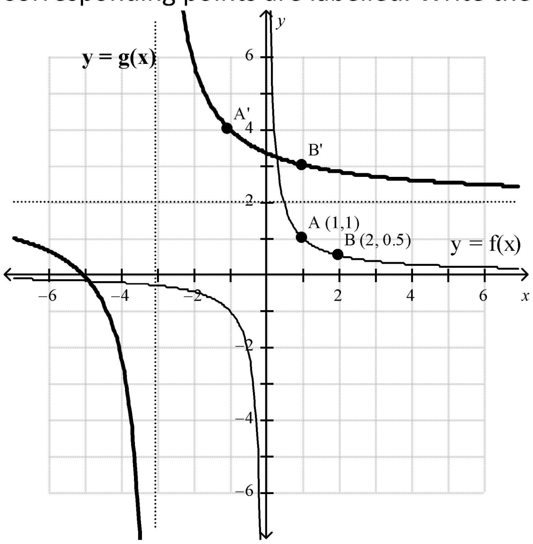


Example 3: The graph of $y = \sqrt{x}$ is vertically compressed by a factor of $\frac{1}{5}$, horizontally compressed by a factor of $\frac{1}{3}$, reflected in the y-axis, then translated 3 units right and 2 units down. Write an equation of the image graph in terms of *x.*

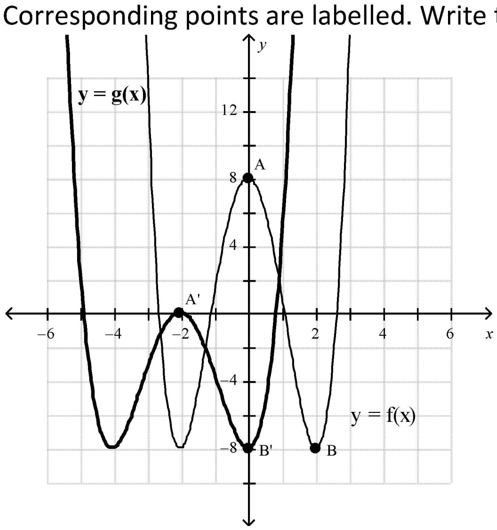
Determining an Equation of a Function After a Transformation

1. Determine two corresponding points on each graph (A and B and A' and B')
2. Determine the horizontal and vertical distance between A and B
3. Determine the horizontal and vertical distance between A' and B'
4. Use this information to determine the a and b values using $\frac{x}{b} = x'$ and $ay = y'$
5. Use one point on both graph, apply stretches, compressions and reflections, and determine the translations using: $\frac{x}{b} + h = x'$ and $ay + k = y'$
6. Substitute all your values into $y - k = af(b(x - h))$

Example 4: The graph of $y = g(x)$ is the image of the graph $y = f(x)$ after a combination of transformations. Corresponding points are labelled. Write then verify an equation for the image graph in terms of the function f.



Example 5: The graph of $y = g(x)$ is the image of the graph $y = f(x)$ after a combination of transformations. Corresponding points are labelled. Write then verify an equation for the image graph in terms of the function f.



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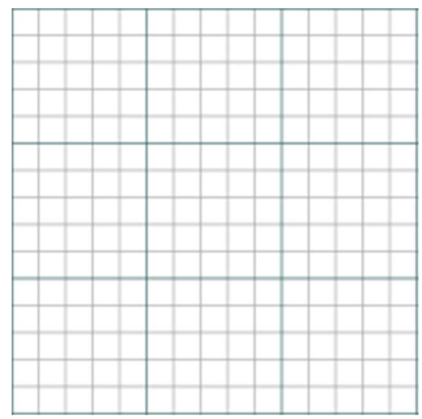
3.5 – Inverse Relations – Part 1

Example 1: Solve for x in the following equations.

a) $y = 3x - 4$ b) $y = \frac{3x-5}{2}$ c) $y = 3x^2 - 5$ d) $y = 2(x-3)^2 + 4$

Inverse – Opposite or "reverse".

Example 2: Graph the function of $y = 2x + 4$ and $y = \frac{1}{2}x - 2$

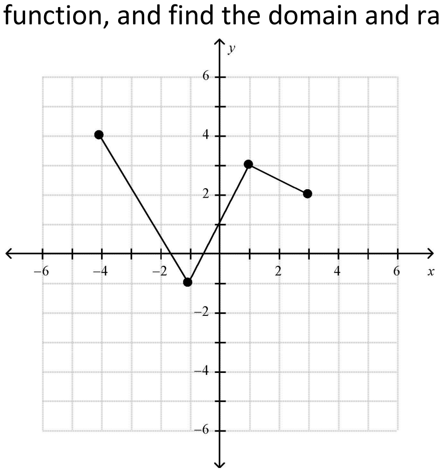


Reflecting in the Line $y = x$ – For a function $y = f(x)$, the graph of $x = f(y)$ is the image of the graph of $y = f(x)$ after a reflection in the line $y = x$. A point (x, y) on $y = f(x)$ corresponds to the point (y, x) on the graph of $x = f(y)$.

Sketching the Inverse of a Function Given a Graph

1. Sketch the line $y = x$
2. Choose points, (x, y), on the line for $y = f(x)$
3. Plot points for $x = f(y)$ as the points (y, x)

Example 3: Here is the graph of y = f(x). Sketch the graph of its inverse on the same graph, determine if it is a function, and find the domain and range of both functions.



If you want another example of this go to example 1 on page 237.

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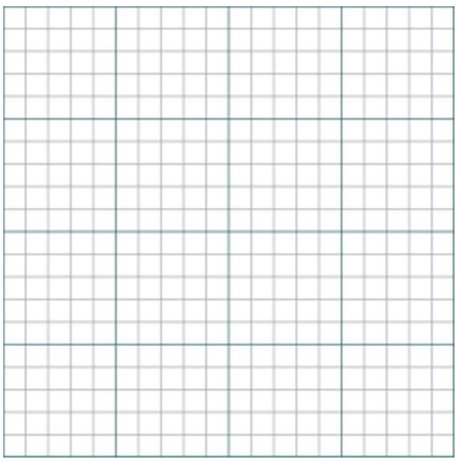
Domain and Range of a Function and its Inverse – The domain of y = f(x) is the range of x = f(y), and the range of y = f(x) is the domain of x = f(y).

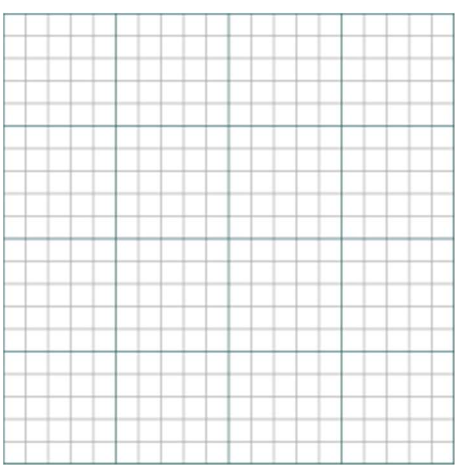
Determining the Inverse Function

1. Interchange x and y in the equation
2. Solve for y

Example 4: Determine an equation of the inverse of $y = x^2 + 3$, sketch the graph as well as its inverse. Determine if the inverse is a function, and state the domain and the range.



Example 5: Determine an equation of the inverse of $y = -x^2 + 4$, sketch the graph as well as its inverse. Determine if the inverse is a function, and state the domain and the range.

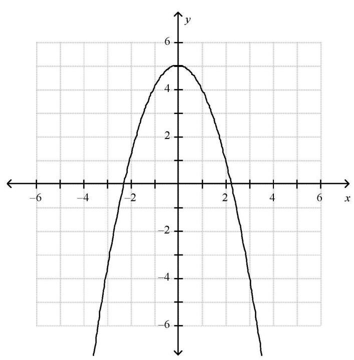


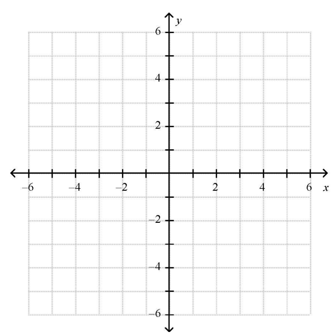
3.5 – Inverse Relations – Part 2

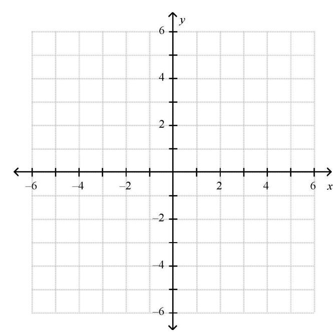
Restrictions on Domain

1. Sketch the inverse of the function
2. Determine if the inverse is a function, if not, determine the point where it doesn't pass the Vertical Line Test
3. Restrict the domain and/or range to only portions of the function that pass the VLT

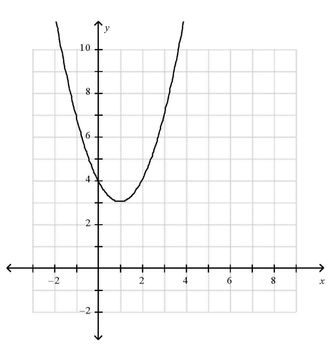
Example 1: Determine two ways to restrict the domain of $y = -x^2 + 5$ so that its inverse is a function. Write the equation of the inverse each time. Use a graph to illustrate each way.

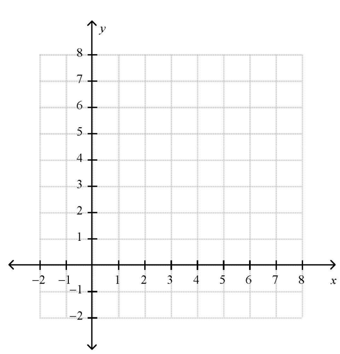


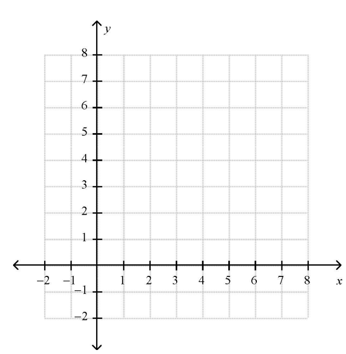




Example 2: Determine two ways to restrict the domain of $y = (x - 1)^2 + 3$ so that its inverse is a function. Write the equation of the inverse each time. Use a graph to illustrate each way.







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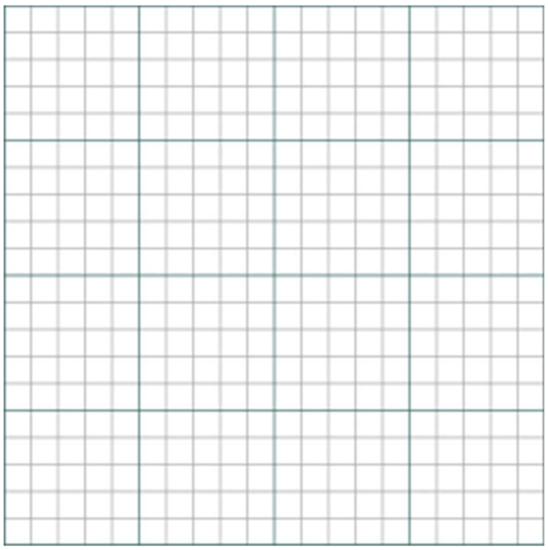
Determining Whether Functions are Inverses

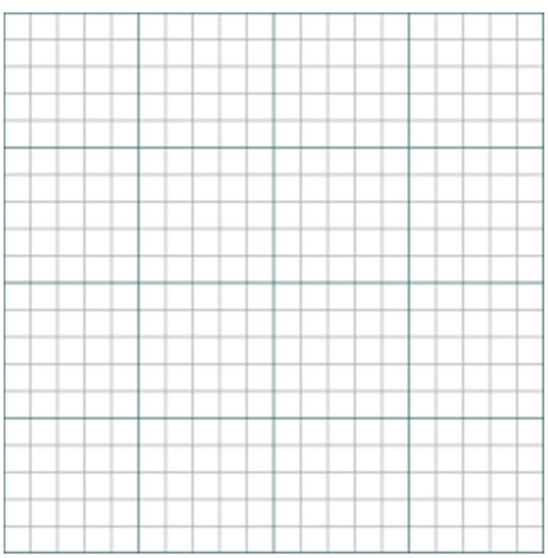
1. Interchange x and y in one equation
2. Solve for y
3. If the equations match, then they are inverses of each other.

Example 3: Determine whether the functions in each pair are inverses of each other algebraically and graphically

a) $y = 3x - 6$ and $y = \frac{x-6}{3}$



b) $y = -x^2 + 3, x \geq 0$ and $y = \sqrt{3 - x}$



If you want another example of this go to example 4 on page 241.